Exam Category Theory

19/01/2012, 14:00-17:00

Please write clearly and start the answer to each question on a new page. You are free to answer in Dutch or English. Write your name on each paper you hand in. There are 4 questions and you can score a total of 20 points.

Question 1 (6 points, in total)
We wish to define a category **Pinj** of sets, and partial injections. The objects of **Pinj** are sets and a morphism $f : X \to Y$ from a set X to a set Y is a partial injection. i.e. f is a subset of $X \times Y$ such that if $(x, y), (x, y') \in f$ then $y = y'$ (f is a partial function) and if $(x, y), (x', y) \in f$ then $x = x'$ (f is injective).

(a) (1 point) Define identities and composition in **Pinj**. (Checking the relevant properties is not required).

(b) (1 point) Let **Rel** be the category of sets and relations between them. Describe the obvious functor $F : \mathbf{Pinj} \to \mathbf{Rel}$ (and check that it is a functor).

(c) (2 points) Is this functor F full? Is it faithful?

Let $\mathbf{C}$ be an arbitrary category. A dagger on $\mathbf{C}$ is a functor $(-)^\dagger : \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$ such that for objects $X \in \mathbf{C}$ we have $X^\dagger = X$ and for morphisms $f : X \to Y$ we have $(f^\dagger)^\dagger = f$.

(d) (2 points) Show that **Pinj** and **Rel** both have daggers. Furthermore show that the functor $F : \mathbf{Pinj} \to \mathbf{Rel}$ preserves the dagger. i.e. satisfies $F(f^\dagger) = F(f)^\dagger$.

Question 2 (3 points)
For a natural number $n \in \mathbb{N}$ define the finite set $\underline{n}$ as object $\underline{n} = \{0, \ldots, n-1\} \in \mathbf{Sets}$. Consider the following diagram in **Sets**, given by the obvious inclusions:

$$\underline{1} \longrightarrow \underline{2} \longrightarrow \underline{3} \longrightarrow \underline{4} \longrightarrow \underline{5} \longrightarrow \cdots$$

Show that $\mathbb{N}$ together with the inclusions $\underline{n} \to \mathbb{N}$ is a colimit of the above diagram.

Question 3 (8 points, in total)
If $\mathbf{A}$ is a category and $F : \mathbf{A} \to \mathbf{A}$ is a functor then the category of algebras of the functor F is defined as follows: the objects of $\mathrm{Alg}(F)$ are arrows $a : FX \to X$ in $\mathbf{A}$. A morphism between two algebras $a : FX \to X$ and $b : FY \to Y$ is a morphism $h : X \to Y$ in $\mathbf{A}$ such that the following diagram commutes:

$$\begin{array}{ccc} FX & \xrightarrow{Fh} & FY \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow[h]{} & Y \end{array}$$

Now suppose we do not only have a functor $F : \mathbf{A} \to \mathbf{A}$ but also functors $G : \mathbf{B} \to \mathbf{B}$ and $H : \mathbf{A} \to \mathbf{B}$ as well as a natural transformation $\alpha : GH \Rightarrow HF$, as in:

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{G} & \mathbf{B} \\ {\scriptstyle H}\uparrow & \overset{\alpha}{\Searrow} & \uparrow{\scriptstyle H} \\ \mathbf{A} & \xrightarrow[F]{} & \mathbf{A} \end{array}$$

(a) (2 points) Show that in this situation there is a functor $\mathrm{Alg}(H) : \mathrm{Alg}(F) \to \mathrm{Alg}(G)$.

Now suppose that $\alpha : GH \Rightarrow HF$ is a natural isomorphism and H has a right adjoint J

$$\begin{array}{c} \mathbf{B} \\ {\scriptstyle H}\uparrow \dashv \downarrow{\scriptstyle J} \\ \mathbf{A} \end{array}$$

Write $\eta : \mathrm{id}_{\mathbf{A}} \Rightarrow JH$ and $\varepsilon : HJ \Rightarrow \mathrm{id}_{\mathbf{B}}$ for the unit and counit of this adjunction $H \dashv J$.

(b) (2 points) Show that there is a natural transformation $\beta : FJ \Rightarrow JG$.

(Hint: take the adjoint transpose of $HFJX \xrightarrow{\alpha^{-1}_{JX}} GHJX \xrightarrow{G\varepsilon_X} GX$)

(c) (4 points) From β and part (a) we get a functor $\mathrm{Alg}(J) : \mathrm{Alg}(G) \to \mathrm{Alg}(F)$; show that it is right adjoint to $\mathrm{Alg}(H)$, in the situation:

$$\begin{array}{c} \mathrm{Alg}(G) \\ {\scriptstyle \mathrm{Alg}(H)}\uparrow \dashv \downarrow{\scriptstyle \mathrm{Alg}(J)} \\ \mathrm{Alg}(F) \end{array}$$

Question 4 (3 points)

Let $T = (\mathbf{C} \xrightarrow{T} \mathbf{C}, \eta, \mu)$ be a monad on a category $\mathbf{C}$. Call T **idempotent** if the mulitplication $\mu\colon T^2 \Rightarrow T$ is an isomorphism. (Aside: this holds, for instance, for the monad describing the completion of a metric space.)

Prove the equivalence of the following three statements.

1. T is idempotent;
2. $T\eta = \eta T$;
3. every algebra of the monad T is an isomorphism.

(Note: here we deal with algebras of a *monad*, not of a *functor*, like in the previous exercise.)