Category Theory – Exercise Sheet 8

November 21, 2013

The deadline is 6pm on Thursday the 28th of November. You can either email your answers to `r.furber` at `cs.ru.nl` (do not forget the attachment) or put them in Robert Furber's postbox on the north side of the second floor of the Huygensgebouw. It is inside a white cabinet with its back to the stairs, opposite the photocopier. Please fasten the sheets of paper together firmly with a staple or paperclip. Folding over the corner is not good enough.

For each question, the number of marks available is indicated in round brackets. The total number is 50.

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For each of the following functors, determine if they are full, faithful, both or neither. No proof is required. (6pt)

1. The forgetful functor $U_1 : \mathbf{Posets} \to \mathbf{Sets}$.
2. The forgetful functor $U_2 : \mathbf{Posets} \to \mathbf{PreOrd}$.
3. The forgetful functor $U_3 : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Ab}$.
4. The constant functor $F : \mathbf{Sets} \to \mathbf{Sets}$ that maps a set to the empty set and a function to the empty function.
5. The constant functor $G : \mathbf{Sets} \to \mathbf{Sets}$ that maps a set to the set $\{1, 2\}$ and a function to the identity.
6. The free monoid functor $(-)^* : \mathbf{Sets} \to \mathbf{Monoids}$.

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Let $F : \mathbf{C} \to \mathbf{D}$ be a full and faithful functor. For each of the following statements determine whether it is true or false. Give a short argument for each part.

(a) If $X \in \mathbf{C}$ is initial then so is $F(X) \in \mathbf{D}$. (2pt)

(b) If $F(X) \in \mathbf{D}$ is initial then so is $X \in \mathbf{C}$. (2pt)

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Show that a functor category $\mathbf{D}^{\mathbf{C}}$ has binary products if $\mathbf{D}$ does. (12pt)
(Hint: Construct the product of two functors F and G "objectwise":
$(F \times G)(C) = F(C) \times G(C)$)

28 Star product

Let $F, G : \mathbf{A} \to \mathbf{B}$ and $H, K : \mathbf{B} \to \mathbf{C}$ be functors.
Let $\eta : F \Rightarrow G$ and $\delta : H \Rightarrow K$ be natural transformations.
For each object A of $\mathbf{A}$, the following diagram commutes by naturality of δ.

$$\begin{array}{ccc} (H \circ F)(A) & \xrightarrow{H(\eta_A)} & (H \circ G)(A) \\ {\scriptstyle \delta_{FA}}\big\downarrow & & \big\downarrow{\scriptstyle \delta_{GA}} \\ (K \circ F)(A) & \xrightarrow[K(\eta_A)]{} & (K \circ G)(A) \end{array}$$

We define the star product $\delta \star \eta$ of the natural transformations δ and η by

$$(\delta \star \eta)_A = \delta_{GA} \circ H(\eta_A) = K(\eta_A) \circ \delta_{FA}$$

for every object $A \in \mathbf{A}$.

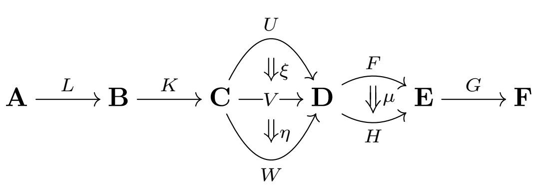
(a) Show that the star product $\delta \star \eta$ forms a natural transformation $H \circ F \Rightarrow K \circ G$. (2pt)

(b) Show that the star product $\star$ is associative. (2pt)

(c) Let $\varepsilon : F \Rightarrow G, \eta : G \Rightarrow H, \mu : K \Rightarrow L, \nu : L \Rightarrow M$ be natural transformations, where $F, G, H : \mathbf{A} \to \mathbf{B}$ and $K, L, M : \mathbf{B} \to \mathbf{C}$ are functors.
Show that $(\nu \circ \mu) \star (\eta \circ \varepsilon) = (\nu \star \eta) \circ (\mu \star \varepsilon)$. (6pt)

29 Godement's five rules

Given the diagram



Prove the following statements:

(a) $1_{G \circ F} \star \xi = 1_G \star (1_F \star \xi)$ and $\xi \star 1_{K \circ L} = (\xi \star 1_K) \star 1_L$ (2pt)

(b) $1_U \star 1_K = 1_{U \circ K}$ and $1_F \star 1_U = 1_{F \circ U}$ (2pt)

(c)
$$\begin{array}{ccc} F \circ U & \xrightarrow{1_F \star \xi} & F \circ V \\ \downarrow{\scriptstyle \mu \star 1_U} & & \downarrow{\scriptstyle \mu \star 1_V} \\ H \circ U & \xrightarrow[1_H \star \xi]{} & H \circ V \end{array}$$
commutes. (2pt)

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For a natural number n, define $\overline{n} = \{0, 1, \ldots, n-1\}$. Let $\mathbf{Sets}_{\text{fin}}$ be the category of finite sets and functions between them and let $\mathbf{N}$ be the full subcategory of $\mathbf{Sets}_{\text{fin}}$ consisting of all $\overline{n}$ for $n \in \mathbb{N}$. Prove in detail that these two categories $\mathbf{Sets}_{\text{fin}}$ and $\mathbf{N}$ are equivalent. (12pt)