Category Theory – Exercise Sheet 6

October 28, 2013

The deadline is 6pm on Thursday the 24th of October. You can either email your answers to `r.furber` at `cs.ru.nl` (do not forget the attachment) or put them in Robert Furber's postbox on the north side of the second floor of the Huygensgebouw. It is inside a white cabinet with its back to the stairs, opposite the photocopier. Please fasten the sheets of paper together firmly with a staple or paperclip. Folding over the corner is not good enough.

For each question, the number of marks available is indicated in round brackets. The total number is 50.

18 Pullbacks

Let $\mathbb{C}$ be an arbitrary category.

(a) Show that an arrow $f : A \to B$ in $\mathbb{C}$ is monic iff (5pt)

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback.

(b) Suppose that the category $\mathbb{C}$ has products and pullbacks. Let $f, g : X \to Y$ be two arrows. Suppose that the following diagram is a pullback: (6pt)

$$\begin{array}{ccc} Z & \xrightarrow{h} & Y \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \langle \mathrm{id}_Y, \mathrm{id}_Y \rangle} \\ X & \xrightarrow[\langle f, g \rangle]{} & Y \times Y \end{array}$$

Prove that e is an equalizer of f and g

(c) Suppose that the category $\mathbb{C}$ has pullbacks. Show that the slice category $\mathbb{C}/A$ has pullbacks. (6pt)

19 Distributivity

Suppose that $\mathbb{C}$ is a category with products $(1, \times)$ and coproducts $(0, +)$. Let A, B, C be objects of the category $\mathbb{C}$.

(a) Show that there is a canonical morphism $(A \times B) + (A \times C) \to A \times (B + C)$ (8pt)

(b) Show that this morphism is always an isomorphism in **Sets**. (5pt)

(c) Prove that this means that the functor $A \times (-) : \mathbf{Sets} \to \mathbf{Sets}$ preserves binary coproducts. (2pt)

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Let $\overrightarrow{n}$ denote the poset of natural numbers $0 \leq 1 \leq \cdots \leq n$ (where $n \in \mathbb{N}$). Consider the sequence of posets $\overrightarrow{0} \to \overrightarrow{1} \to \cdots$, where each arrow is an inclusion.

Determine the limit and colimit posets of this sequence, in the category of posets and monotone functions. (12pt)

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Prove that in category with exponentials, there is an isomorphism $C^{A \times B} \cong (C^A)^B$. Describe the maps in both directions explicitly and prove that they are each other's inverse. (6pt)