Category Theory – Exercise Sheet 5

October 11, 2013

The deadline is 6pm on Thursday the 17th of October. You can either email your answers to `r.furber` at `cs.ru.nl` (do not forget the attachment) or put them in Robert Furber's postbox on the north side of the second floor of the Huygensgebouw. It is inside a white cabinet with its back to the stairs, opposite the photocopier. Please fasten the sheets of paper together firmly with a staple or paperclip. Folding over the corner is not good enough.

For each question, the number of marks available is indicated in round brackets. The total number is 50.

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The *discrete category* on a set I is the category with I as its set of objects and only identity morphisms. A (possibly infinite) product in a category $\mathbb{C}$ is a limit of a diagram whose shape is a discrete category. More explicitly, the product of an I-indexed family of objects $\{D_i\}$ where $D_i \in \mathrm{Obj}(\mathbb{C})$ is an object

$$\prod_{i\in I} D_i$$

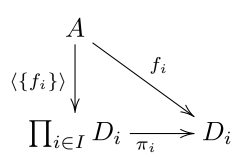
with an I-indexed family of projection mappings $\{\pi_i\}_{i\in I}$ where

$$\pi_j : \prod_{i\in I} D_i \to D_j$$

This object and its family of projection maps are required to satisfy a UMP, which is that for every object A and I-indexed family of maps $\{f_i\}_{i\in I}$, such that $f_i : A \to D_i$ we have that there is a unique map

$$\langle\{f_i\}\rangle : A \to \pi_{i\in I} D_i$$

such that for every $i \in I$ the following triangle commutes



If you cannot see this, it may help to expand the definition to show it agrees with the usual binary and finite products if I is a 2-element or finite set respectively.

(a) Show that products in a poset are given by greatest lower bounds, if they exist. (3pt)

(b) Coproducts are defined in the dual way. Show that if a poset has products it has coproducts. (There is no need to reprove the dual of part (a).) (4pt)

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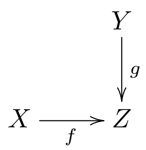
In the following, let $\mathbb{C}$ have all finite limits. (16pt)
Recall the definition of the slice category $\mathbb{C}/Z$ for a category $\mathbb{C}$ and an object Z in that category. Show that, for each pair of maps with common codomain



their pullback in $\mathbb{C}$ is a product in $\mathbb{C}/Z$ and vice-versa.

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(a) Let P be a poset. Characterize the maps that are equalizers in P. (4pt)

(b) Show that every monic is an equalizer in **Sets**. (5pt)

(c) Give an example of a category with a monic that is not an equalizer. You may find it helpful to use exercise 8 (b). (2pt)

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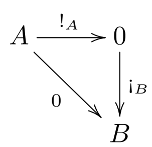
A *zero object* in $\mathbb{C}$ is an object that is both initial and terminal. In any category with a zero object 0, there is a *zero map* $0 : A \to B$ between any two objects A, B given as follows:



i.e. $0 = ¡_B \circ !_A$.

(a) In **Monoids**, describe the zero object and zero maps, proving the properties. In $\mathbf{Vect}_{\mathbb{R}}$, give the zero object and zero maps (you do not need to reprove that they are what you say they are). (7pt)

(b) In each category with zero objects, the "predicate" kernel of $f : A \to B$ is defined as the equalizer

$$\ker(f) \longrightarrow A \underset{0}{\overset{f}{\rightrightarrows}} B$$

if it exists.

Describe $\ker(f)$ in **Monoids** and $\mathbf{Vect}_{\mathbb{R}}$. Again, give the proof for **Monoids** but only the result in $\mathbf{Vect}_{\mathbb{R}}$. (7pt)

(c) In a category with products, we can define the "relation" kernel of a map $f : A \to B$ as the equalizer

$$\ker(f) \longrightarrow A \times A \underset{f \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} B$$

if it exists.

Describe $\ker(f)$ in **Sets**, in particular the relation it defines between elements of A. It may help you to use the expression for the equalizer in **Sets** given in the lectures. (2pt)