Category Theory – Exercise Sheet 4

October 2, 2013

The deadline is 6pm on Thursday the 10th of October. You can either email your answers to `r.furber` at `cs.ru.nl` (do not forget the attachment) or put them in Robert Furber's postbox on the north side of the second floor of the Huygensgebouw. It is inside a white cabinet with its back to the stairs, opposite the photocopier. Please fasten the sheets of paper together with a staple or paperclip.

For each question, the number of marks available is indicated in round brackets. The total number is 50.

12 Playing with Products

Let $\mathbf{C}$ be a category with products.

1. Show that for all objects A_1, A_2, A_3 in $\mathbf{C}$ there is a canonical isomorphism (4pt)

$$\alpha_{A_1,A_2,A_3} \colon A_1 \times (A_2 \times A_3) \to (A_1 \times A_2) \times A_3$$

such that for all $f_i \colon A_i \to B_i$ one has

$$\alpha_{B_1,B_2,B_3} \circ f_1 \times (f_2 \times f_3) = ((f_1 \times f_2) \times f_3) \circ \alpha_{A_1,A_2,A_3}.$$

You may leave out the subscripts.

2. Show that for all objects A in $\mathbf{C}$ there is a canonical arrow $\Delta_A \colon A \to A \times A$ that "copies" A. Prove that for each morphism $f \colon A \to B$ one has $\Delta_B \circ f = (f \times f) \circ \Delta_A$. (2pt)

3. Let us assume that $\mathbf{C}$ has also a terminal object 1. Recall from the last sheet that for all objects A in $\mathbf{C}$ there are isomorphisms $\sigma_{l,A} \colon A \to A \times 1$ and $\sigma_{r,A} \colon A \to 1 \times A$.

   Show that for the unique arrow $! \colon 1 \to 1$ and any morphism $f \colon A \to B$ we have that $\sigma_r \circ f = (f \times !) \circ \sigma_r$. The analogous result holds for σ_l. (2pt)

13 C-Monoids and Their Limits

Let **C** be a category with products and a terminal object 1. We are going to define monoids in **C** and organise them into a category. Then we will show that this category has limits, provided **C** has them.

Using the isomorphism $\alpha_A \colon A \times (A \times A) \to (A \times A) \times A$ constructed in Exercise 12 we can define what it means for an arrow $m \colon A \times A \to A$ in **C** to be *associative*, namely if the following diagram commutes

$$\begin{array}{ccccc} A \times (A \times A) & & \xrightarrow{\alpha_A} & & (A \times A) \times A \\ {\scriptstyle id_A \times m}\big\downarrow & & & & \big\downarrow{\scriptstyle m \times \mathrm{id}_A} \\ A \times A & \xrightarrow{m} & A & \xleftarrow{m} & A \times A \end{array} \quad . \qquad \text{(Assoc)}$$

Using the isomorphisms $\sigma_{r,A} \colon A \to A \times 1$ and $\sigma_{l,A} \colon A \to 1 \times A$ constructed on the last sheet, then an arrow $e \colon 1 \to A$ is a *unit* for m, if

$$\begin{array}{ccccc} A \times 1 & \xleftarrow{\sigma_{r,A}} & A & \xrightarrow{\sigma_{l,A}} & 1 \times A \\ {\scriptstyle \mathrm{id}_A \times e}\big\downarrow & & \big\| & & \big\downarrow{\scriptstyle e \times \mathrm{id}_A} \\ A \times A & \xrightarrow{m} & A & \xleftarrow{m} & A \times A \end{array} \qquad \text{(Unit)}$$

commutes.

Using these definitions we call a triple

$$(A, m \colon A \times A \to A, e \colon 1 \to A)$$

consisting of an object A and arrows in **C** a **C**-*monoid* if (Assoc) and (Unit) commute, i.e. m is associative and e is its unit.

(a) Instantiate this definition in **Sets** and show that the definition of a **Sets**-monoid coincides with the usual definition. (6pt)

(b) Let (A_1, m_1, e_1) and (A_2, m_2, e_2) be **C**-monoids and $f \colon A_1 \to A_2$ an arrow in **C**. Define what it means for f to be a **C**-monoid *homomorphism*. Then show that this coincides with the usual definition of monoid homomorphisms in **Sets**. (6pt)

(c) Show that for all **C**-monoids (A, m, e) and (A_i, m_i, e_i) with $i = 1, 2, 3$

  (i) $\mathrm{id}_A \colon A \to A$ is a **C**-monoid homomorphism. (2pt)

  (ii) If $f \colon A_1 \to A_2$ and $g \colon A_2 \to A_3$ are **C**-monoid homomorphisms, then $g \circ f$ is one. (4pt)

By (c) we know that we can organise **C**-monoids and their homomorphisms into a category **Mon**(**C**).

(d) Show that the category **Mon**(**C**) has binary products. (12pt)
Hint: You have to define the monoid operation point-wise. Prove associativity and the unit axiom component-wise and start by drawing the corresponding diagrams, say (Assoc), of the component in the middle. Then proceed by drawing (Assoc) for the product around and it connect it to the inner diagram in a way that everything commutes.

(e) Show that if **C** has equaliser, the category **Mon**(**C**) has them, too. (12pt)