Category Theory Exercise Sheet 2

September 18, 2013

The deadline is 6pm on Thursday the 26th of September. You can either email your answers to `r.furber` at `cs.ru.nl` or put them in the Robert Furber's postbox on the north side of the second floor of the Huygensgebouw. It is inside a white cabinet with its back to the stairs, opposite the photocopier.

For each question, the number of marks available is indicated in round brackets. The total number is 50.

5 Functors

For many categories (called locally small in the book) the (co)-slice can be represented in the category of sets. Let $\mathbf{C}$ be a category and denote the set of arrows between $A, B \in |\mathbf{C}|$ by $\mathrm{Hom}_{\mathbf{C}}(A, B)$. We write for simplicity $\mathrm{Hom}_{\mathbf{C}}(A, -)(B) = \mathrm{Hom}_{\mathbf{C}}(A, B)$ and $\mathrm{Hom}_{\mathbf{C}}(-, B)(A) = \mathrm{Hom}_{\mathbf{C}}(A, B)$ respectively.

(a) Show that for all object $A \in \mathbf{C}$ the mapping $\mathrm{Hom}_{\mathbf{C}}(A, -)$ can be extended to a functor $\mathbf{C} \to \mathbf{Sets}$. (4)

(b) Show that for all object $A \in \mathbf{C}$ the mapping $\mathrm{Hom}_{\mathbf{C}}(-, A)$ can be extended to a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Sets}$. (4)

6 Universal properties

Let P be a set together with an operation $\vee\colon P \times P \to P$, written as $a \vee b$ for elements $a, b \in P$, which is associative, commutative and idempotent (meaning that $a \vee a = a$). This operation is called *join*. Let us assume furthermore that there is a unit element $\bot \in P$ for the join: $p \vee \bot = p = \bot \vee p$ for all $p \in P$. We call such a triple $(P, \vee, \bot)$ a *join-semilattice* (JSL).

Let **JSL** be the category of join-semilattices and homomorphisms, where a homomorphism $f\colon (P_1, \vee_1, \bot_1) \to (P_2, \vee_2, \bot_2)$ of JSLs is a map $f\colon P_1 \to P_2$ between the underlying sets which preserves the join and unit: $f(a \vee_1 b) = f(a) \vee_2 f(b)$ and $f(\bot_1) = \bot_2$. The category **JSL** is equipped with an obvious forgetful functor $U\colon \mathbf{JSL} \to \mathbf{Sets}$ given by $U(P, \vee, \bot) = P$ and $U(f) = f$.

Now let X be an arbitary set and consider $\mathcal{P}_{\text{fin}}(X)$ the set of finite subsets of X:

$$\mathcal{P}_{\text{fin}}(X) = \{A \subseteq X \mid A \text{ finite}\}\,.$$

(a) Show that $\mathcal{P}_{\text{fin}}$ extends to a functor $\mathcal{P}_{\text{fin}}\colon \mathbf{Sets} \to \mathbf{JSL}$. (6)

(b) Show that there is an inclusion map $\eta_X\colon X \to U\left(\mathcal{P}_{\text{fin}}(X)\right)$ which is universal in the following way. For all join-semilattices $(P, \vee, \bot)$ and maps $f\colon X \to P$ there is a unique arrow $\bar{f}\colon \mathcal{P}_{\text{fin}}(X) \to (P, \vee, \bot)$ in $\mathbf{JSL}$ such that (7)

$$\begin{array}{ccc} U(\mathcal{P}_{\text{fin}}(X)) & \xrightarrow{U(\bar{f})} & P \\ {\scriptstyle \eta_X}\uparrow & \nearrow{\scriptstyle f} & \\ X & & \end{array}$$

commutes.
You may use here that for all join-semilattice $(P, \vee, \bot)$ and all finite subsets $A \subseteq P$ the join $\bigvee A$ exists and is given by $\bigvee \emptyset = \bot$ and $\bigvee\{a_1, \dots, a_n\} = a_1 \vee \cdots \vee a_n$.

7 Constructions on Categories

Let I be a fixed set of indices. Let $\overline{\mathbf{Sets}^I}$ be the category of

- I-indexed families $\{X_i\}_{i\in I}$ of sets with $X_i \cap X_j = \emptyset$ if $i \neq j$ and
- morphisms $f\colon \{X_i\}_{i\in I} \to \{Y_i\}_{i\in I}$ given as family $\{f_i\}_{i\in I}$ of maps with $f_i\colon X_i \to Y_i$ composed component-wise.

Show that this category is isomorphic to the slice category $\mathbf{Sets}/I$, i.e. there is an isomorphism $\mathbf{Sets}/I \cong \overline{\mathbf{Sets}^I}$ of categories. (14)

8 Mono-, Epi- and Isomorphisms

(a) Consider the following commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & C \end{array}$$

in any category $\mathbf{C}$. Show

(i) if f and g are monos (epis), so is h. (2)

(ii) if h is a mono, so is f. (2)

(iii) if h is an epi, so is g. (2)

(iv) if h is a mono, g doesn't have to be monic. Give a counterexample. (2)

(b) Let $(P, \leq)$ be a poset and consider the associated poset category $\mathbf{P}$ with $x \in P$ as objects. Show that every morphism in $\mathbf{P}$ is both epic and monic. What are the isos? Conclude that an arrow which is epi and mono doesn't have to be an iso. (3)

We say that a functor $F\colon \mathbf{C} \to \mathbf{D}$ *preserves* isomorphisms (monos, epis) if for all isomorphisms (monos, epis) $f\colon X \to Y$ in $\mathbf{C}$ its image $F(f)$ is an iso (mono, epi) in $\mathbf{D}$.

(c) Show that every functor preserves isomorphisms. (2)

(d) Let $\mathbf{C}$ be a category and A an object in $\mathbf{C}$. Show that the hom-functor $\mathrm{Hom}_{\mathbf{C}}(A, -)\colon \mathbf{C} \to \mathbf{Sets}$ preserves monos. (2)

References