Category Theory – Exercise Sheet 13

January 9, 2014

The deadline is 6pm on Thursday the 16th of January. You can either email your answers to `r.furber` at `cs.ru.nl` (do not forget the attachment) or put them in Robert Furber's postbox on the north side of the second floor of the Huygensgebouw. It is inside a white cabinet with its back to the stairs, opposite the photocopier. Please fasten the sheets of paper together firmly with a staple or paperclip. Folding over the corner is not good enough.

For each question, the number of marks available is indicated in round brackets. The total number is 50.

42

In exercise 37 a functor $F : \mathbf{Sets} \to \mathbf{Vect}_{\mathbb{R}}$ was described, the left adjoint to the forgetful functor. We shall now consider the resulting monad on **Sets**, also written F.

(a) Write down the definition of the unit maps $\eta_X : X \to F(X)$ of this monad. Using this, the definition of the adjunction in exercise 37 and the extension of F to a functor defined using exercise 36, give a formula for $F(f)(a)(y)$ where $f : X \to Y$ in **Sets**, $a \in F(X)$ and $y \in Y$. You are not required to reprove that F is a functor and η a natural transformation. (6pt)

(b) Describe the multiplication $\mu_X : F^2(X) \to F(X)$. You are not required to prove naturality, but you must show that $\mu_X \circ F(\eta_X) = \mathrm{id}_{FX} = \mu_X \circ \eta_{F(X)}$. (11pt)

(c) Consider the Kleisli category $\mathcal{K}\ell(F)$. Show that for objects $m, n \in \mathbb{N}$, considered as finite sets, maps $m \to n$ in $\mathcal{K}\ell(F)$ correspond to $m \times n$ matrices. Which is to say, define an isomorphism between $\mathcal{K}\ell(F)(m, n)$ and the set of $m \times n$ matrices, $M_{m\times n}$. (5pt)

(d) Prove that Kleisli composition of $A : m \to n$ and $B : n \to p$ in $\mathcal{K}\ell(F)$ corresponds to matrix multiplication. (8pt)

43

Let A be a fixed set. Consider the definition $F(X) = 1 + A \times X$ of a functor $\mathbf{Sets} \to \mathbf{Sets}$.

(a) Describe what F does on functions. (2pt)

(b) Consider the set A^* of finite lists of elements of A. Define the "obvious" F-algebra $a : F(A^*) \to A^*$ given by "nil" and "cons", i.e. the empty list ϵ and prefixing an element to a list. (3pt)

(c) Show that this algebra $a : F(A^*) \to A^*$ is an initial object in the category $\mathbf{Alg}(F)$. (15pt)