Category Theory – Exercise Sheet 12

December 17, 2013

The deadline is 6pm on Thursday the 9th of January. You can either email your answers to `r.furber` at `cs.ru.nl` (do not forget the attachment) or put them in Robert Furber's postbox on the north side of the second floor of the Huygensgebouw. It is inside a white cabinet with its back to the stairs, opposite the photocopier. Please fasten the sheets of paper together firmly with a staple or paperclip. Folding over the corner is not good enough.

For each question, the number of marks available is indicated in round brackets. The total number is 50.

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Let F be a functor with two right adjoints: $F \dashv G$ with natural transformations η, ϵ and $F \dashv G'$ with natural transformations η', ϵ'.

(a) Use these units and counits to explicitly construct maps $GX \to G'X$ and $G'X \to GX$. (4pt)

(b) Show that these maps are natural transformations. (4pt)

(c) Show that these maps are each other's inverses. Conclude that $G \cong G'$. (4pt)

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Let $\mathbf{C}$ be a category with finite products. An internal monoid in $\mathbf{C}$ consists of an object $M \in \mathbf{C}$ along with two maps $m : M \times M \to M$ and $u : 1 \to M$, called the multiplication and unit respectively, such that the following diagrams commute:

$$\begin{tikzcd} 1 \times M \arrow[r, "u\times \mathrm{id}"] \arrow[dr, "\simeq"'] & M \times M \arrow[d, "m"] & M \times 1 \arrow[l, "\mathrm{id}\times u"'] \arrow[dl, "\simeq"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} M \times M \times M \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times \mathrm{id}"'] & M \times M \arrow[d, "m"] \\ M \times M \arrow[r, "m"'] & M \end{tikzcd}$$

Prove in detail that $M \times (-) : \mathbf{C} \to \mathbf{C}$ is a monad (8pt)

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Show that the Eilenberg-Moore category of the powerset monad $\mathcal{P}$ on **Sets** is equivalent to the category of complete lattices, by following the steps below.

(a) (i) Given a complete lattice L, consider the map $\bigvee : \mathcal{P}L \to L$ and show that this is an Eilenberg-Moore algebra of the powerset monad. (6pt)

(ii) Given a join-preserving map $f : L \to K$ between complete lattices, prove that f is a map of Eilenberg-Moore algebras of the powerset monad, that is, the following diagram commutes: (5pt)

$$\begin{tikzcd} \mathcal{P}L \arrow[r, "\mathcal{P}f"] \arrow[d, "\bigvee"'] & \mathcal{P}K \arrow[d, "\bigvee"] \\ L \arrow[r, "f"] & K \end{tikzcd}$$

(b) Let $\alpha : \mathcal{P}X \to X$ be an Eilenberg-Moore algebra

(i) We now define a relation on X by $x \leq y \iff \alpha(\{x, y\}) = y$. Check that this is a partial order. Hint: for the transitivity of the relation $\leq$, use the multiplication law of the powerset monad. (8pt)

(ii) Show that the set X is a complete lattice, with $\alpha = \bigvee$. (6pt)

(iii) Show that the Eilenberg-Moore algebra maps of the powerset monad preserve these joins. (5pt)

You now have two functors $\mathbf{CL}_\vee \to \mathcal{EM}(\mathcal{P})$ and $\mathcal{EM}(\mathcal{P}) \to \mathbf{CL}_\vee$. It is easy to check that they are each other's inverses.