Category Theory – Exercise Sheet 11

December 12, 2013

The deadline is 6pm on Thursday the 19th of December. You can either email your answers to `r.furber` at `cs.ru.nl` (do not forget the attachment) or put them in Robert Furber's postbox on the north side of the second floor of the Huygensgebouw. It is inside a white cabinet with its back to the stairs, opposite the photocopier. Please fasten the sheets of paper together firmly with a staple or paperclip. Folding over the corner is not good enough.

For each question, the number of marks available is indicated in round brackets. The total number is 50.

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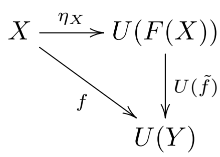
Let $U : \mathcal{C} \to \mathcal{D}$ be a functor, and $F : \mathrm{Obj}(\mathcal{D}) \to \mathrm{Obj}(\mathcal{C})$ a function. Suppose there is an $\mathrm{Obj}(\mathcal{D})$-indexed family $\{\eta_X\}$ of $\mathcal{D}$-maps such that $\eta_X : X \to U(F(X))$, and for every $f : X \to U(Y)$ in $\mathcal{D}$, there is a unique $\tilde{f} : F(X) \to Y$ in $\mathcal{C}$ such that the following diagram commutes



Define F on maps to make a functor such that $\{\eta_X\}$ defines a natural transformation. (14pt)

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Let $\mathbf{Vect}_{\mathbb{R}}$ be the category of $\mathbb{R}$-vector spaces and $\mathbb{R}$-linear maps. We have a forgetful functor $U : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Sets}$ which takes the underlying set of each vector space and for each linear map $f : V \to W$ takes the underlying function. For each set X we may define $F(X)$ as follows

$$F(X) = \{a : X \to \mathbb{R} | \mathrm{supp}(a) \text{ is finite}\}$$

where $\text{supp}(a) = \{x \in X | a(x) \neq 0\}$. This can be given a vector space structure in the following way (where $a, b \in F(X)$, $x \in X$ and $\alpha \in \mathbb{R}$):

$$(a + b)(x) = a(x) + b(x)$$
$$(\alpha \cdot f)(x) = \alpha f(x)$$

You are not required to verify the vector space axioms for $F(X)$.

Using the previous exercise, show that the assignment $F(X)$ can be extended to a functor $\mathbf{Sets} \to \mathbf{Vect}_{\mathbb{R}}$ that is the left adjoint of U. (18pt)

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Consider two complete lattices X and Y as (small) categories, and a monotone map $g : X \to Y$ as a functor between them.

Show that if $g : X \to Y$ preserves limits, then g has a left adjoint $f : Y \to X$, given by the formula (18pt)

$$f(x) = \bigwedge g^{-1}(\uparrow a)$$

Show also that if g preserves colimits, it has a right adjoint $h : Y \to X$, and give the formula for h.

Hint: You may want to use exercises 14(a) and 16(a) from sheet 5 to understand what limits and colimits are in posets. You may reuse the result showing naturality of the hom-set bijection is automatic for posets from exercise 33 on the previous sheet.

Remark (for those interested): The analogous theorem is false for complete and cocomplete categories, as well as large posets. See [1, V.6 page 123] and [2].

References

[1] Saunders Mac Lane. *Categories for the Working Mathematician.* Graduate Texts in Mathematics. Springer Verlag, 1971.

[2] Robert M. Solovay. New Proof of a Theorem of Gaifman and Hales. *Bulletin of the American Mathematical Society*, 72:282–284, 1966.