Category Theory – Exercise Sheet 10

December 5, 2013

The deadline is 6pm on Thursday the 12th of December. You can either email your answers to `r.furber` at `cs.ru.nl` (do not forget the attachment) or put them in Robert Furber's postbox on the north side of the second floor of the Huygensgebouw. It is inside a white cabinet with its back to the stairs, opposite the photocopier. Please fasten the sheets of paper together firmly with a staple or paperclip. Folding over the corner is not good enough.

For each question, the number of marks available is indicated in round brackets. The total number is 50.

Note: You may choose yourself which formulation of the concept of adjunction you want to use, but you do have to prove the naturality conditions involved.

33 Adjoints and functions

For a set A, we consider the poset $(\mathcal{P}(A), \subseteq)$ as a category.
Given a function between sets $f : A \to B$, we define the three following functors:

- The direct image functor $\amalg_f : \mathcal{P}(A) \to \mathcal{P}(B)$ defined by

  $$\amalg_f(X) = \{b \in B \mid \exists x \in X.\ b = f(x)\}$$

  for every $X \subseteq A$.

- The inverse image functor $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ defined by

  $$f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$$

  for every $Y \subseteq B$.

- The functor $\Pi_f : \mathcal{P}(A) \to \mathcal{P}(B)$ defined by

  $$\Pi_f(X) = \{b \in B \mid f^{-1}(\{b\}) \subseteq X\}$$

  for every $X \subseteq A$.

1. Show that $\amalg_f \dashv f^{-1}$ holds for every function f between sets. Check for yourself that the naturality conditions trivially hold in this poset case. (12pt)

2. Show that $f^{-1} \dashv \Pi_f$ holds for every function f between sets. (12pt)

3. Show that in the special case of a projection $\pi_1 : A \times B \to A$, the left and right adjoint to π_1^{-1} can be understood as existential quantification $\exists$ and universal quantification $\forall$ (Hint: a set $X \subseteq A \times B$ can be interpreted as a proposition about elements of $A \times B$). (6pt)

34 Powerset functor

Show that the contravariant powerset functor $\mathcal{P} : \mathbf{Sets}^{\mathrm{op}} \to \mathbf{Sets}$ is self-adjoint (i.e. it has itself as a left adjoint). (6pt)

35 Complete lattices and posets

Let $\mathbf{CL}_\vee$ be the category of complete lattices and join-preserving maps. Prove that the forgetful functor $U : \mathbf{CL}_\vee \to \mathbf{Posets}$ has a left adjoint F given by (14pt)

$$F(X, \leq) = \{U \subseteq X \mid \forall x, y \in X, y \leq x \in U \implies y \in U\}$$

ordered by inclusion.