Category Theory Exercise Sheet 1

September 10, 2013

The deadline is 6pm on Thursday the 19th of September. You can either email your answers to `r.furber` at `cs.ru.nl` or put them in the Robert Furber's postbox on the north side of the second floor of the Huygensgebouw. It is inside a white cabinet with its back to the stairs, opposite the photocopier.

For each question, the number of marks available is indicated in round brackets. The total number is 50.

1

In this question, you will prove some basic facts about isomorphisms that are valid in any category. When proving two expressions are equal, give full details using the axiomatic definition of a category.

(a) Let $f : X \to Y$ be a map. If $g, g' : Y \to X$ are both inverses of f, then $g = g'$.

(3)

(b) Show that $\mathrm{id}_X : X \to X$ is an isomorphism.

(2)

(c) Show that if $f : X \to Y, g : Y \to Z$ are isomorphisms, $g \circ f$ is an isomorphism.

(5)

(d) $\mathrm{Aut}(X)$ is the set of isomorphisms $X \to X$. What algebraic structure does it have?

(2)

2 (Awodey 1.9.3)

Show that

(a) In **Sets**, the isomorphisms are exactly the bijections.

(7)

(b) In **Monoids**, the isomorphisms are exactly the bijective homomorphisms.

(4)

(c) In **Posets**, the isomorphisms are not the same as the bijective homomorphisms.

(6)

3

(a) Show that a monoid homomorphism $f : \mathbb{N} \to \mathbb{Z}$ is determined by $f(1)$.

(2)

(b) Show that such an f can never have an inverse (in **Monoids**).

(4)

4 [1, page 9]

The arrows in a category form a structure in their own right. We can define it as follows.

Consider a set A, with a relation $R \subseteq A \times A \times A$. We write the relation $R(f, g, h)$ as $f \bullet g = h$, and we say that (f, g) is *composable* iff there exists an h such that $f \bullet g = h$. An identity element of A is a $u \in A$ such that $f \bullet u = f$ whenever (f, u) is composable and $u \bullet g = g$ whenever (u, g) is composable.

We define an *arrows-only-category* as a pair $(A, R \subseteq A \times A \times A)$ such that:

(i) If $f \bullet g = h$ and $f \bullet g = h'$ then $h = h'$.

(ii) $(k \bullet g) \bullet f$ is defined iff $k \bullet (g \bullet f)$ is defined, and $(k \bullet g) \bullet f = k \bullet (g \bullet f)$ in this case.

(iii) If (k, g) and (g, f) are composable, then $(k \bullet g, f)$ (equivalently $(k, g \bullet f)$) is composable.

(iv) For each $g \in A$, there exist identity arrows $u, u' \in A$ such that (u, g) and (g, u') are composable.

4.1 Questions

(a) Show that the arrows of every category make an arrows-only-category.

(5)

(b) Show that the identity arrows in an arrows-only-category are unique, and that an identity arrow is equal to its own identity arrows.

(5)

(c) Show that from every arrows-only-category one can recover the objects to make a category.

(5)

References

[1] Saunders Mac Lane. *Categories for the Working Mathematician.* Graduate Texts in Mathematics. Springer Verlag, 1971.